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The macrodynamics of α-effect dynamos in rotating fluids

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Past study of the large-scale consequences of forced small-scale motions in electrically conducting fluids has led to the 'α-effect' dynamos. Various linear kinematic aspects of these dynamos have been explored, suggesting their value in the interpretation of observed planetary and stellar magnetic fields. However, large-scale magnetic fields with global boundary conditions can not be force free and in general will cause large-scale motions as they grow. In this paper the finite amplitude behaviour of global magnetic fields and the large-scale flows induced by them in rotating systems is investigated. In general, viscous and ohmic dissipative mechanisms both play a role in determining the amplitude and structure of the flows and magnetic fields which evolve. In circumstances where ohmic loss is the principal dissipation, it is found that determination of a geostrophic flow is an essential part of the solution of the basic stability problem. Nonlinear aspects of the theory include flow amplitudes which are independent of the rotation and a total magnetic energy which is directly proportional to the rotation. Constant α is the simplest example exhibiting the various dynamic balances of this stabilizing mechanism for planetary dynamos. A detailed analysis is made for this case to determine the initial equilibrium of fields and flows in a rotating sphere.

1. Introduction

Many large-scale features of stellar and planetary magnetic fields may have their origin in rotational constraints acting directly on the large-scale flows, and may be insensitive to the detailed structure of the underlying small-scale motions responsible for magnetic regeneration. It is therefore instructive to investigate the effects of these constraints, and in this study an exploration is begun of the finite amplitude balance of large-scale fields and flows in a rotating sphere based on the ideas of 'mean-field electrodynamics' (Steenbeck, Krause & Rädler 1966; Krause & Steenbeck 1967; Steenbeck & Krause 1969*a*, *b*; Rädler 1968; for further references see Roberts & Stix 1971). This approach, which is related to earlier work by Parker (1955) and Braginskii (1964), supposes that the magnetic and velocity fields in a conducting fluid exist on two widely

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differing length scales, l and L, say, where $l \ll L$. Then the induction equation for the large-scale or mean magnetic field $\mathbf{B}$ may be written as

$$\partial \mathbf{B}/\partial t = \nabla \wedge (\mathbf{U} \wedge \mathbf{B}) + \nabla \wedge (\overline{\mathbf{U}' \wedge \mathbf{B}'}) + \lambda \nabla^2 \mathbf{B}, \tag{1.1}$$

where $\mathbf{U}$ is the mean velocity and λ the magnetic diffusivity of the fluid, and $\mathbf{U}'$ and $\mathbf{B}'$ are the small-scale velocity and magnetic fields. The bar denotes an average over a volume large compared with l^3. The principal problem addressed in the past work has been to express the interaction term $\overline{\mathbf{U}' \wedge \mathbf{B}'}$ in terms of the mean fields; a recent study by Moffatt (1970*a*, *b*, 1972) supposes that the small-scale motions take the form of quasi-linear inertial waves and finds that to a first approximation

$$(\overline{\mathbf{U}' \wedge \mathbf{B}'})_i = \alpha f_{ij} B_j + \ldots, \tag{1.2}$$

where α is constant and f_{ij} is a non-dimensional symmetric tensor function of position which depends on the spectrum of the mean 'helicity' ($\equiv \overline{\mathbf{U}' . \nabla \wedge \mathbf{U}'}$) of the small-scale flow. Equation (1.2) is known as the 'α-effect'. It can be shown that for α sufficiently large regeneration is possible. αf_{ij} also depends, at finite amplitude, on $\mathbf{B}$ and $\mathbf{U}$ and in his 1972 paper Moffatt finds a finite amplitude equilibration (in a homogeneous unbounded medium) based on this dependence.

An alternative mechanism which can restrict the growth of magnetic fields is the concomitant development of large-scale velocity fields. In a bounded domain, such as the earth's core, the growing magnetic field will give rise to large-scale Lorentz forces which in general are not irrotational. The dynamical equation for the large-scale flows in an incompressible rotating fluid is written as

$$\partial \mathbf{U}/\partial t + \mathbf{U} . \nabla \mathbf{U} - (\mu\rho)^{-1} \mathbf{B} . \nabla \mathbf{B} + 2\mathbf{\Omega} \wedge \mathbf{U} = -\nabla p + \nu \nabla^2 \mathbf{U} - \nabla . \mathbf{R} + \mathbf{F}, \tag{1.3}$$

where $\mathbf{R}$ is the Reynolds stress due to the small-scale fields, $\mathbf{\Omega}$ is the angular velocity of the frame of reference, ν is the kinematic viscosity, μ is the magnetic permeability, ρ is the density, p is the effective pressure and $\mathbf{F}$ is an arbitrary large-scale body force. In order to isolate the stabilizing mechanism proposed here, $\mathbf{F}$ and $\nabla . \mathbf{R}$ are chosen either equal or zero. Conditions permitting the latter choice are discussed in the conclusion. Hence we visualize an equilibration caused by an increased dissipation rate, primarily ohmic, and due to distortions of the magnetic field resulting from the induced velocity field.

Geophysical observations suggest that the Coriolis and Lorentz forces in the core are of the same order of magnitude. It is assumed here that all the terms in the magnetic diffusion equation (1.1) are of the same order of magnitude and that the characteristic time is that for ohmic decay. The resulting scaled version of the dynamical equation (1.3) contains small terms representing the advection of momentum and viscous dissipation. When both of these small terms can be neglected, the necessary and sufficient condition for solutions of (1.3) is that

$$\int_{C(s)} (\mathbf{B} . \nabla \mathbf{B})_\phi \, d\phi \, dz \equiv 0, \tag{1.4}$$

where $C(s)$ is the cylinder of radius s inscribed in the domain coaxial with $\mathbf{\Omega}$. This condition that there be no 'geostrophic' torques is due to Taylor (1963).

Here it leads to an unusual type of eigenvalue problem for the magnetic fields. The particular problem tackled in §§2–4 was first formulated by Childress (1969) in a slightly different context and with a different aim in view. See pages 424 and 428 below.

Unfortunately it is not possible to neglect viscous effects at small field amplitudes. The initial equilibration of the growing magnetic field is due to the (small) viscous losses. The desirability of establishing the formal link between the 'viscous regime' (for which (1.4) is incorrect) and the 'inviscid regime' (for which (1.4) is central to the problem) was pointed out by a referee of a first draft of this manuscript. The behaviour of equilibrium solutions in the transition region between these two regimes has been resolved by one of us (M.R.E.P.) and is outlined in §5 of this paper.

In §2 we seek steady or periodic axisymmetric solutions to (1.1) and (1.3) in a sphere of radius L for an axisymmetric and isotropic f_{ij} (together with

$$\nabla . \mathbf{U} = \nabla . \mathbf{B} = 0$$

and boundary conditions appropriate to an external insulator). The system is supposed to be near to the state in which energy fed in from the small scales via the α-effect is only just enough to maintain the field, that is, near to the infinitesimal amplitude state in which α is the eigenvalue of a linear eigenvalue problem for given $f_{ij} = f(r, \theta)\, \delta_{ij}$. The nonlinear problem is then attacked by the methods of modified perturbation theory. The conditions for the neglect either of (1.4) (the 'viscous' limit) or of viscosity (the 'inviscid' limit) are set out, and in §§3 and 4 relationships between α and the field amplitude are found for each limit in the special case $f = 1$, for which the underlying eigenvalue problem is of well-known form. Section 4 concludes with a discussion of the novel eigenvalue problem that arises for general f_{ij} in the inviscid limit, where a zonal flow has to be determined as part of the eigensolution. An evaluation is made of the importance of these zonal flows in the published numerical studies of kinematic α-effect dynamos. The viscous–inviscid transition region is discussed in §5. In the concluding section, §6, the generality of the equilibration mechanism explored here is assessed and work in progress is described.

2. Formulation of the problem

2.1. *Scaling and decomposition of the equations*

We seek solutions of (1.1) and (1.3) in a sphere of radius L such that $\mathbf{B}$ is steady or oscillatory in time.

Our first task is to adopt a scaling which makes Lorentz and Coriolis forces of the same order, in line with our discussion in §1. We also presume all the terms in (1.1) to be of the same order, so that L^2/λ is the characteristic time scale. These requirements lead to the unique non-dimensionalization

$$\mathbf{r} = L\hat{\mathbf{r}}, \quad \mathbf{\Omega} = \Omega\hat{\mathbf{k}}, \quad \mathbf{U} = \frac{\lambda}{L}\hat{\mathbf{U}}, \quad \alpha = \frac{\lambda}{L}\hat{\alpha}, \quad t = \frac{L^2}{\lambda}\hat{t}, \quad p = \Omega\lambda\hat{p}, \quad \mathbf{B} = (\Omega\lambda\mu\rho)^{\frac{1}{2}}\hat{\mathbf{B}}.$$

Substituting into (1.1)–(1.3) and dropping hats, we obtain

$$E_M(\partial \mathbf{U}/\partial t + \mathbf{U}.\nabla\mathbf{U}) + 2\mathbf{k} \wedge \mathbf{U} = -\nabla p + \mathbf{B}.\nabla\mathbf{B} + E\nabla^2\mathbf{U}, \quad (2.1a)$$

$$\partial \mathbf{B}/\partial t = \nabla \wedge (\mathbf{U} \wedge \mathbf{B}) + \alpha\nabla \wedge (\mathbf{f}.\mathbf{B}) + \nabla^2\mathbf{B}, \quad (2.1b)$$

$$\nabla.\mathbf{U} = \nabla.\mathbf{B} = 0 \quad (2.1c)$$

for $|\mathbf{r}| \leqslant 1$. Here $E \equiv \nu/\Omega L^2$ and $E_M \equiv \lambda/\Omega L^2$ are the Ekman and 'magnetic Ekman' numbers of the large-scale processes. We suppose that $\lambda \rightarrow \infty$ for the material outside the sphere, corresponding to an insulator. Then in this region $\mathbf{U} \equiv 0$ and $\nabla \wedge \mathbf{B} \equiv 0$, with $\mathbf{U}$ and $\mathbf{B}$ continuous across $|\mathbf{r}| = 1$, and $|\mathbf{B}| = O(|\mathbf{r}|^{-3})$ as $|\mathbf{r}| \rightarrow \infty$. We have dropped the Reynolds stress and the large-scale forcing term that appear in (2.1). This is equivalent to supposing that

$$|\nabla.\mathbf{R}|, |\mathbf{F}| \ll \Omega\lambda/L.$$

We discuss this supposition in the conclusion.

$E, E_M \ll 1$ in all situations of interest (e.g. in the earth $10^{-14} < E < 10^{-6}$, $E_M = O(10^{-7})$; see Roberts & Soward 1972). Hence $\Omega^{-1}E_M^{-\frac{1}{2}}$, the Alfvén time scale of the system, is much smaller than the ohmic decay time $\Omega^{-1}E_M^{-1}$ and so it seems reasonable to suppose that energy propagation via Alfvén waves is effectively instantaneous. We therefore neglect the terms in E_M, which suppresses these waves, but makes the constraint (1.4) necessary for solutions to exist.

Although the term in E is small almost everywhere, the question of the neglect of this term depends on whether the magnetic field amplitude is so large that the effect of the viscous boundary layers can be neglected in comparison with that of the Taylor constraint (1.4). This is a delicate question and will be discussed further when the finite amplitude expansion has been developed. For the present, we place the $E\nabla^2\mathbf{U}$ term in brackets to indicate that it may be neglected away from boundary layers.

Before proceeding, we make the simplifying, but physically plausible assumption that f_{ij} is independent of ϕ, where ϕ is the longitude in a spherical (r, θ, ϕ) or cylindrical (z, s, ϕ) co-ordinate system with $\mathbf{k}$ defining the axis of symmetry. In addition, we shall restrict this exploration to a study of axisymmetric solutions (but see the discussion in §3.1) so that the magnetic field may be split into poloidal and toroidal parts. Following Bullard & Gellman (1954), we may write any solenoidal axisymmetric vector field $\mathbf{Y}$ as

$$\mathbf{Y} = y\hat{\mathbf{e}}_\phi + \nabla \wedge (x\hat{\mathbf{e}}_\phi), \quad (2.2)$$

where $\hat{\mathbf{e}}_\phi$ is the unit vector in the ϕ direction. The first and second terms on the right-hand side are called the toroidal and poloidal parts of $\mathbf{Y}$, respectively. Note that these two vectors are orthogonal. If we now write

$$\mathbf{B} = b\hat{\mathbf{e}}_\phi + \nabla \wedge (a\hat{\mathbf{e}}_\phi) \quad \text{and} \quad \nabla \wedge (\mathbf{U} \wedge \mathbf{B}) = d\hat{\mathbf{e}}_\phi + \nabla \wedge (c\hat{\mathbf{e}}_\phi)$$

we may equate poloidal and toroidal parts in (2.1b), and 'uncurl' the poloidal equation since the electrostatic field $-\nabla\Phi$ has no ϕ component. We then obtain

$$\left.\begin{aligned} \partial a/\partial t &= c + \alpha f b + D^2 a, \\ \partial b/\partial t &= d + \alpha\hat{\mathbf{e}}_\phi.\nabla \wedge [f\nabla \wedge (a\hat{\mathbf{e}}_\phi)] + D^2 b, \end{aligned}\right\} \quad (2.3)$$

where $D^2 \equiv \nabla^2 - 1/s^2$ and we have made the analytically convenient simplification $f_{ij} = f\delta_{ij}$, which we do not anticipate to have more than a qualitative effect on our results. We henceforward use these equations together with the equation for $\mathbf{U}$,

$$2\mathbf{k} \wedge \mathbf{U} = -\nabla p + \mathbf{B}.\nabla\mathbf{B} + [E\nabla^2\mathbf{U}]. \tag{2.4}$$

2.2. *The finite amplitude expansion*

We now make use of the methods of modified perturbation theory (e.g. Malkus & Veronis 1958); that is, we suppose that the amplitude of the magnetic field is $O(\epsilon) \ll 1$, and that $\Delta\alpha \equiv \alpha - \alpha_0 = O(\epsilon)$, where α_0 is the eigenvalue obtained as $|\mathbf{B}| \to 0$. We then expand all quantities in powers of ϵ, supposing that $t = \tau/q$, so that $\partial/\partial t = q\,\partial/\partial\tau$ (Veronis 1959):

$$x = \sum_{n=0}^{\infty} \epsilon^n x_n, \tag{2.5}$$

where $x = \mathbf{B}, \mathbf{U}, a, b, c, d, \alpha, q$ or p and where $b_0 = a_0 = \mathbf{B}_0 = 0$. We seek solutions proportional to $e^{i\tau}$. We may then equate powers of ϵ and solve a sequence of linear problems. The arbitrariness in ϵ is removed by an appropriate normalization condition on b_1; we set

$$\{b_1^+ b_1^*\} = 1, \quad \{b_1^+ b^*\} = \epsilon, \tag{2.6}$$

where curly brackets denote an integral over the sphere $|\mathbf{r}| \leqslant 1$ and an asterisk denotes a complex conjugate; b_1^+ is the adjoint of b_1 with respect to the linear operator L_0 [see equation (2.13) below]. b_1 is used as the basis of the expansion since it vanishes for $|\mathbf{r}| \geqslant 1$ as do all the b_j. It is supposed, as seems reasonable, that the eigenfunction b_1 is unique (however, see the discussion in §3.1). The boundary conditions on a_j are

$$D^2 a_j = 0 \quad \text{for} \quad |\mathbf{r}| \geqslant 1, \quad a_j, \partial a_j/\partial r \quad \text{continuous across} \quad |\mathbf{r}| = 1, \tag{2.7}$$

so that any choice other than b_1 as a basis for the expansion would involve integrals over all space, whereas choosing b_1 ensures that integrals may be taken only over the domain of interest. We also note that (2.6) implies that

$$\{b_1^+ b_j^*\} = 0, \quad j \neq 1. \tag{2.8}$$

We may now write down the problem sequence†

$$2\mathbf{k} \wedge \mathbf{U}_0 = -\nabla p_0 + [E\nabla^2\mathbf{U}_0], \tag{2.9a}$$

$$0 = -iq_0 a_1 + (\mathbf{U}_0 \wedge \mathbf{B}_1).\hat{\mathbf{e}}_\phi + \alpha_0 f b_1 + D^2 a_1, \tag{2.9b}$$

$$0 = -iq_0 b_1 + \nabla \wedge (\mathbf{U}_0 \wedge \mathbf{B}_1).\hat{\mathbf{e}}_\phi + \alpha_0 \hat{\mathbf{e}}_\phi.\nabla \wedge [f\nabla \wedge (a_1 \hat{\mathbf{e}}_\phi)] + D^2 b_1, \tag{2.9c}$$

and for $j \geqslant 2$

$$2\mathbf{k} \wedge \mathbf{U}_j = -\nabla p_j + \sum_{i=1}^{j} \operatorname{Re}(\mathbf{B}_i).\nabla \operatorname{Re}(\mathbf{B}_{j-i}) + [E\nabla^2\mathbf{U}_j], \tag{2.10a}$$

† We note that $\mathbf{U}_{2N+1}$, α_{2N+1} and q_{2N+1} and $\mathbf{B}_{2N}$ may be taken as zero without any loss of generality, since the nonlinearity of the equations is cubic, so that structure is added only at every other order in ϵ.

$$0 = \sum_{i=1}^{j} (-iq_{j-i}a_i + \alpha_{j-i}fb_i) + c'_j + (\mathbf{U}_0 \wedge \mathbf{B}_j).\hat{\mathbf{e}}_\phi + \alpha_0 fb_j + D^2 a_j + \text{c.c.} + \text{non-secular terms}, \quad (2.10b)$$

$$0 = \sum_{i=1}^{j} [-iq_{j-i}b_i + \alpha_{j-i}\hat{\mathbf{e}}_\phi.\nabla \wedge [f\nabla \wedge (a_i\hat{\mathbf{e}}_\phi)]] + d'_j + \nabla \wedge (\mathbf{U}_0 \wedge \mathbf{B}_j).\hat{\mathbf{e}}_\phi + \alpha_0\hat{\mathbf{e}}_\phi.\nabla \wedge [f\nabla \wedge (a_j\hat{\mathbf{e}}_\phi)] + D^2 b_j + \text{c.c.} + \text{non-secular terms},$$

where c.c. denotes the complex conjugate. There are further terms in the higher-order magnetic equations that do not have $e^{\pm i\tau}$ time dependence and hence cannot give rise to any secularities; for the purpose of finding α_{j-1} and q_{j-1} we only need the parts of a_j, b_j etc. which could give rise to secular behaviour.

We have

$$c'_j = c_j - (\mathbf{U}_0 \wedge \mathbf{B}_j).\hat{\mathbf{e}}_\phi,$$

$$d'_j = d_j - \nabla \wedge (\mathbf{U}_0 \wedge \mathbf{B}_j).\hat{\mathbf{e}}_\phi,$$

with c_j and d_j given by (2.3). We first note that (2.9) constitutes an eigenvalue problem (α_0 and q_0 being the eigenvalues) for $\mathbf{B}_1$. The form of $\mathbf{U}_0$ depends on the importance of viscosity in relation to the constraints on the field implied by the Taylor condition (see below, §2.3). We suppose that $\mathbf{U}_0$ is known, and (2.9a) has the solution

$$\mathbf{U}_0 = V_0(s)\,\hat{\mathbf{e}}_\phi \quad (2.11)$$

away from boundary layers, when the boundary conditions are taken into account. Hence

$$c'_j = c_j, \quad d'_j = d_j + G_0(s)\frac{\partial a_j}{\partial z}, \quad G_0(s) = s\frac{d}{\partial s}\left(\frac{V_0}{s}\right). \quad (2.12)$$

Note that $\mathbf{U}_j$ for $j \geqslant 2$ will consist of a part driven by Lorentz forces and a 'free' part $V_j(s)\,\hat{\mathbf{e}}_\phi$ analogous to $V_0(s)\,\hat{\mathbf{e}}_\phi$.

We may now define the linear operator L_0 (in principle at least). Equation (2.9b) may be used to express a_1 in terms of b_1 (this can be done more or less straightforwardly by means of Green's functions). If the result is substituted into (2.9c), we obtain the single linear integro-differential equation

$$L_0(b_1) = 0, \quad (2.13)$$

where $L_0 = L_0(\alpha_0, q_0, G_0(s))$. In the same way (2.11) becomes

$$L_0(b_j) + F_j + \text{c.c.} = 0. \quad (2.14)$$

F_j is a combination of inhomogeneous terms, and we have retained only those parts of F_j and b_j (and their complex conjugates) which have $e^{\pm i\tau}$ dependence. All the terms in F_j are known except α_{j-1} and q_{j-1} and we choose these to eliminate the secularities in (2.14). That is, if we define b_1^+ as the adjoint of b_1 with respect to L_0, so that

$$\left.\begin{aligned} \{b_1^+ L_0^*(b^*)\} &\equiv b\{L_0^+(b_1^+)\}, \\ L_0^+(b_1^+) &= 0, \end{aligned}\right\} \quad b_1^+ \propto e^{i\tau},$$

then $\{b_1^+ L_0^*(b_j^*)\} = 0$. Now $F_j + F_j^* = F_1 e^{i\tau} + F_2 e^{-i\tau}$, so to eliminate the secularities we multiply the $e^{i\tau}$ equation by b_1^{+*} and the $e^{-i\tau}$ equation by b_1^+ and integrate, so that

$$\{F_1 b_1^{+*}\} = \{F_2 b_1^+\} = 0, \tag{2.15}$$

and this fixes the unknowns α_{j-1} and q_{j-1}. It is now possible to solve for a_j and b_j and continue the process. In the event that $q_0 \neq 0$ and $V_0 \neq 0$, the eigensolutions will not be proportional to $e^{i\tau}$ since $V_0 = V_0(\tau)$. However, when the eigensolutions are known, α_j etc. may still be found by a similar analysis. It is practical to carry through this programme only for a very restricted class of f's. Here, we shall complete the analysis in the special case $f = 1$, for which L_0 is self-adjoint and takes a particularly simple form. When we have found α_j and q_j, to any required level, we may invert the expansion for α in (2.5) and obtain a relation between ϵ and $\Delta\alpha$.

2.3. *The viscous and inviscid limits*

We now give a brief statement of the conditions for the neglect or otherwise of viscosity. The justification for these conditions is contained in §5, when the structure of the finite amplitude problem has been made clear.

(i) If E is bounded away from zero and $\epsilon \to 0$, so that $\epsilon^2 E^{-\frac{1}{2}} \to 0$, then the effect of Ekman suction reduces $V_j(s)$ to zero, so that only the driven part of $\mathbf{U}_j$ remains (the 'viscous limit').

(ii) If ϵ is bounded away from zero, and E is so small that $\epsilon^{-2}E^{\frac{1}{2}} \ll \epsilon^2$, then the effect on $\mathbf{U}_j$ of Ekman suction is unimportant compared with the requirement that $\mathbf{B}.\nabla\mathbf{B}$ satisfies the Taylor condition (1.4) at all orders in ϵ. The free part $V_j(s)$ of $\mathbf{U}_j$ is determined by (1.4) at the $j+2$ level (see §4; the 'inviscid' limit).

We treat case (i) in §3 and case (ii) in §4. (Although case (i) leads to easier mathematics, it seems that the region of ϵ, E space relevant in astrophysical contexts is that for which case (ii) is appropriate.) In §5, the transition region between the two limits is investigated, and it is shown how the two limits are connected. The reader wishing to understand the equilibration mechanism without becoming immersed in boundary-layer calculations may omit the last part of §3 and §5, since the remainder is self-contained. Note that it is necessary to consider the inviscid limit in order to justify the scaling and ordering, since our ordering implies that ohmic, and not viscous, dissipation is important in the final equilibrium.

2.4. *Discussion*

We must show, to justify the expansion scheme, that $\alpha_j = O(1)$ for some $j \neq 1$; we do this for the special case $f = 1$, which is the only one readily accessible to analytical methods, and contains most of the interesting physics. In §4, we discuss more general f's, since in the inviscid limit, the eigenvalue problem depends crucially on the parity of f with respect to $z = 0$. We end this section with some consequences of the scaling and expansion which are independent of the form of f.

(i) The induced velocity fields have magnitudes $O(\lambda/L)$, independent of Ω.

(ii) The magnetic energy is proportional to $\Omega\lambda$. (This is in contrast to Moffatt's (1972) equation (6.3), which gives a magnetic energy proportional to $\Omega^{-\frac{1}{3}}$.)

(iii) The ratio of kinetic to magnetic energy is $O(E_M) \ll 1$, so that the system does not approach equipartition.

These general consequences could, we hope, be used to provide some observational corroboration (or otherwise) of the results that follow.

3. Solution for the case $f = 1$ in the viscous limit

3.1. *The eigenvalue problem*

For $f = 1$, the magnetic equations in (2.9) become

$$\left.\begin{aligned} 0 &= -iq_0 a_1 + \alpha_0 b_1 + D^2 a_1, \\ 0 &= -iq_0 b_1 - \alpha_0 D^2 a_1 + D^2 b_1, \end{aligned}\right\} \quad |\mathbf{r}| \leqslant 1, \tag{3.1}$$

with $D^2 a_1 \equiv b_1 \equiv 0$, $|\mathbf{r}| \geqslant 1$, and b_1, a_1 and $\partial a_1/\partial r$ continuous across $|\mathbf{r}| = 1$; $|a_1| = O(|\mathbf{r}|^{-2})$ as $|\mathbf{r}| \to \infty$. $\mathbf{U}_0$ is identically zero in the viscous limit.

We seek solutions of (3.1) such that α_0 is a minimum. Our first remark is that the principle of exchange of stabilities is valid, as shown in appendix A, and we therefore set $q_0 = 0$, and seek steady perturbations from a steady solution. However, there could exist non-axisymmetric wavelike instabilities of the solutions; these waves would be akin to those discussed by Hide (1966), Malkus (1967) and Acheson (1972, 1972) and may be important in explaining the geomagnetic secular variation. We do not discuss these waves here.

We note that the rotation axis $\mathbf{k}$ does not enter the problem at this stage; we shall take the axis of symmetry of b_1 and a_1 to be $\mathbf{k}$† so that we may eliminate any ϕ dependence at finite amplitude. This eliminates the possibility that a tilted magnetic field may be preferred at finite amplitude (as observed in the earth), but a study of tilt is beyond the scope of this paper.

Since $q_0 = 0$, the equations become

$$\left.\begin{aligned} 0 &= \alpha_0 b_1 + D^2 a_1, \\ 0 &= -\alpha_0 D^2 a_1 + D^2 b_1, \end{aligned}\right\} \quad |\mathbf{r}| \leqslant 1, \tag{3.2}$$

and in this case we find that the operator L_0 takes the very simple form

$$L_0 \equiv D^2 + \alpha_0^2;$$

L_0 is now self-adjoint, and we notice that the determination of a_1 and $\mathbf{B}_1$ can be deferred until after the determination of b_1. This problem has been solved by Childress (1969) and Krause & Steenbeck (1967); here we follow the treatment of the latter.

The solutions for the lowest mode take the form

$$\left.\begin{aligned} b_1 &= B r^{-\frac{1}{2}} J_{\frac{3}{2}}(\alpha_0 r) \sin\theta \\ a_1 &= B/\alpha_0 \left(r^{-\frac{1}{2}} J_{\frac{3}{2}}(\alpha_0 r) \sin\theta - \tfrac{1}{3}\alpha_0 r J_{\frac{1}{2}}(\alpha_0) \sin\theta\right) \end{aligned}\right\} \quad \text{for} \quad |\mathbf{r}| \leqslant 1; \tag{3.3}$$

† This ensures that b_1 is unique to within a constant.

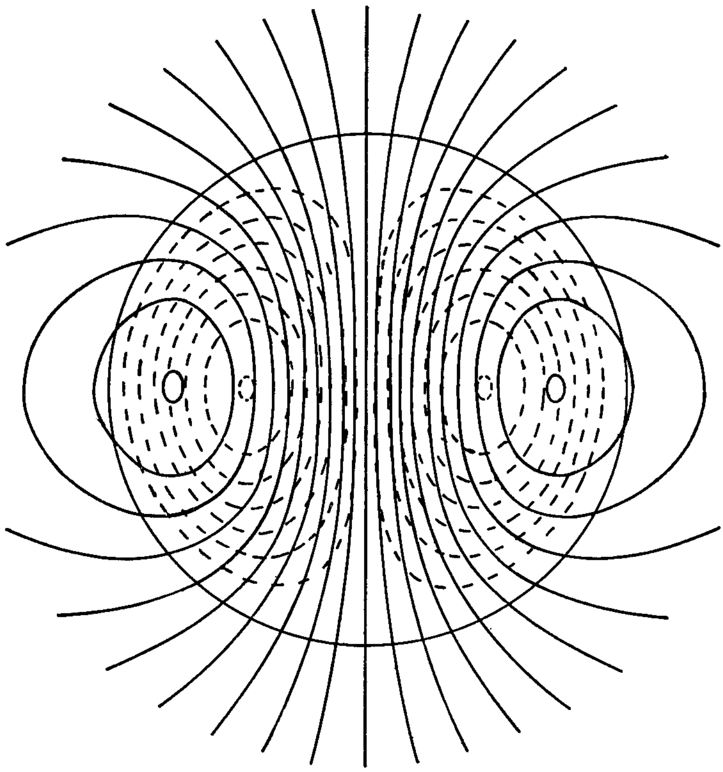

FIGURE 1. The magnetic field $\mathbf{B}_1$ from (3.3). ———, poloidal lines of force; – – –, lines of toroidal field strength (after Krause & Steenbeck).

for $|\mathbf{r}| \geqslant 1$, $b_1 = 0$ and $a_1 = (-B/3r^2)\,J_{\frac{1}{2}}(\alpha_0)\sin\theta$. α_0 is the smallest zero of $J_{\frac{3}{2}}(x)$; equivalently

$$\alpha_0 = \tan\alpha_0 \simeq \pm 4{\cdot}49.$$

Figure 1, from Krause & Steenbeck (1967), exhibits this solution. Because L_0 is self-adjoint the problem for α_0 can be expressed in variational form; α_0^2 is the minimum of the functional

$$F[b] \equiv \{|\nabla b|^2 + |b|^2/s^2\}/\{|b|^2\}, \tag{3.4}$$

where b is any sufficiently well behaved function of r and θ that vanishes at $r = 0, 1$, as can be seen by manipulating $\{b(\alpha_0 b + D^2 b)\}$ and using the divergence theorem. We make use of this result in §4. B is a normalization factor defined by (2.6) above, which is equivalent to $\{b_1^2\} = 1$. Hence

$$B^2 = 3(\alpha_0^2+1)/4\pi|\alpha_0| \quad \text{if} \quad J_{\frac{3}{2}}(x) \equiv x^{-\frac{3}{2}}\sin x - x^{-\frac{1}{2}}\cos x$$
$$= 1{\cdot}13\ldots\,.$$

3.2. *The determination of* α_2

We have not had to concern ourselves with viscous effects up to now, since no velocities appear in the eigenvalue problem. Before we can proceed to finite amplitude, however, we should take account of the effect of the $E\nabla^2\mathbf{U}$ term, which will lead to a thin layer of high shear adjacent to the boundary. Following Greenspan (1968), we suppose that all the fields can be split into interior and boundary-layer parts; that is, we set, for example,

$$\mathbf{U}_i = \mathbf{U}_{i0} + E^{\frac{1}{2}}\mathbf{U}_{i1} + \ldots + \tilde{\mathbf{U}}_{i0} + E^{\frac{1}{2}}\tilde{\mathbf{U}}_i + \ldots, \tag{3.5}$$

and similarly for $\mathbf{B}_i$, etc., where $\tilde{\mathbf{U}}_{ij}$ is a function of a boundary-layer co-ordinate $\zeta \equiv E^{-\frac{1}{2}}(1-r)$, and $\tilde{\mathbf{U}}_{ij} \to 0$ as $\zeta \to \infty$. These boundary-layer terms provide a transition from the inviscid boundary condition $\mathbf{U}_i . \mathbf{n} = 0$ to the viscous one $\mathbf{U}_i = 0$ at $\zeta = 0$. The choice of $E^{\frac{1}{2}}$ as an expansion parameter is natural since the boundary layer is just a modified Ekman layer. (There is no magnetic diffusion layer since we are considering steady solutions, and the Hartmann number $M \equiv |\mathbf{B}|\, L/(\mu\rho\nu\lambda)^{\frac{1}{2}} = E^{-\frac{1}{2}}$ from the scaling of §2, so the Hartmann thickness M^{-1} and Ekman thickness are of the same order.) We anticipated this scaling in §2 when we discussed the viscous and inviscid limits. We also expand

$$\alpha_j = \alpha_{j0} + E^{\frac{1}{2}}\alpha_{ji} + \ldots$$

for each j.

We now have a two-parameter expansion scheme, in powers of ϵ^2 and $E^{\frac{1}{2}}$, and it is clear that the ratio of these two quantities is of vital importance in determining the order in which we should tackle the higher-level equations. [When we are considering situations between the two extremes, a third parameter $\eta \equiv \epsilon^2 E^{-\frac{1}{2}}$ becomes important, but there seems to be no point in further complicating the picture by introducing it at this stage, especially since it replaces one or other of ϵ^2 and $E^{\frac{1}{2}}$ when it is used. η is used in §5 to describe the balance between Ekman suction and Lorentz torques which prevails in intermediate ranges.]

When $E^{\frac{1}{2}} \gg \epsilon^2$, the most important correction to α_{00} due to finite amplitude and/or viscosity should be $\alpha_{01}E^{\frac{1}{2}}$, but this is zero since $\mathbf{B}_1$ has no boundary-layer structure. (Clearly, $\mathbf{B}_1$ has no velocities associated with it and hence has no reason to be affected by viscosity.) The first change in α that might not be zero is $\alpha_{20}\epsilon^2$; to find this, we need the relevant equations at order E^0, which are

$$2\mathbf{k} \wedge \mathbf{U}_{20} = -\nabla p_{20} + \mathbf{B}_{10} . \nabla\mathbf{B}_{10}, \tag{3.6a}$$

$$0 = c_{30} + \alpha_{20} b_{10} + \alpha_{00} b_{30} + D^2 a_{30}, \tag{3.6b}$$

$$0 = d_{30} - \alpha_{20} D^2 a_{10} - \alpha_{00} D^2 a_{30} + D^2 b_{30}, \tag{3.6c}$$

where the term in E is now neglected. The boundary condition on $\mathbf{U}_{20}$ is

$$\mathbf{U}_{20} . \mathbf{n} = 0$$

(Greenspan 1968, p. 42). The boundary condition on b_{30} is $b_{30}|_{r=1} = 0$ since (as will be shown below) $\tilde{b}_{30}(\zeta) \equiv 0$ ($b_{10} \equiv b_1$ as defined by (3.3), with $\alpha_{00} = \alpha_0$). The equation analogous to (2.14) is

$$0 = \alpha_{00} c_{30} + d_{30} + 2\alpha_{00}\alpha_{20} b_{10} + L_0(b_{30}), \tag{3.7}$$

where $L_0 \equiv D^2 + \alpha_{00}^2$. Note that $c'_{j0} = c_{j0}$ and $d'_{j0} = d_{j0}$ since $\mathbf{U}_{00} \equiv 0$. The solvability condition (2.15) can now be written as

$$\alpha_{20} = (-2\alpha_{00})\,\{b_{10}(\alpha_{00} c_{30} + d_{30})\}. \tag{3.8}$$

It is an easy task to show that this expression in fact vanishes. We note that a_{10} and b_{10} are even functions of z, and hence that B_{10s} is odd and B_{10z} is even. This in turn, using (3.6*a*), leads to the conclusion that U_{20s} is odd and U_{20z} and $U_{20\phi}$ are even. We therefore finally arrive at the conclusion that $c_{30} \equiv (\mathbf{U}_{20} \wedge \mathbf{B}_{10})_\phi$ and $d_{30} \equiv [\nabla \wedge (\mathbf{U}_{20} \wedge \mathbf{B}_{10})]_\phi$ are both odd functions of z, and hence that the integral in (3.8) vanishes. Unfortunately, although this argument is useful when the integrals are odd functions of z, it tells us nothing about any integrals of even functions that may arise. Another and easier way to see that $\alpha_{20} = 0$ is to make use of an identity which not only determines (3.8) but which will be essential to the determination of α_{40}. It is

$$\{bd\} \equiv \{cD^2a\}, \tag{3.9}$$

which holds for all a, b, c and d that satisfy (2.4) with inviscid boundary conditions and viscosity neglected. This is proved in appendix B. The importance of this identity is that no surface integral terms appear in it, as they would in the equivalent identity if the problem had been set up in terms of $\mathbf{B}_1$ rather than b_1. This is the advantage that the decomposition of § 2 has over a vector formulation, since we shall be able to express α_{40} without the need to use cumbersome surface integrals of undetermined sign. Meanwhile we note that (3.9) implies that at $O(\epsilon^4)$

$$\begin{aligned} 0 &= \{b_{10}d_{30} - c_{30}D^2a_{10}\} \\ &= \{b_{10}d_{30} + \alpha_{00}b_{10}c_{30}\} \quad \text{from (3.2), since} \quad c_{10} = d_{10} = 0, \end{aligned}$$

and so the result $\alpha_{20} = 0$ is simply recovered.

The next coefficient in the expansion of α that we should try to evaluate is α_{21}, since $\epsilon^2E^{\frac{1}{2}} \gg \epsilon^4$ in the limit considered. This will involve the solution of equations for the boundary-layer terms $\tilde{\mathbf{U}}_{20}$ and $\tilde{\mathbf{B}}_{31}$ ($\tilde{\mathbf{B}}_{30} \equiv 0$, as we shall see), and so we must first determine $\mathbf{U}_{20}$.

3.3. *The determination of* $\mathbf{U}_{20}$

From (3.6) we have

$$2\mathbf{k} \wedge \mathbf{U}_{20} = -\nabla p'_{20} + (\nabla \wedge \mathbf{B}_{10}) \wedge \mathbf{B}_{10}, \tag{3.10}$$

where $\mathbf{B}_{10}.\nabla\mathbf{B}_{10}$ has been replaced by an alternative form, with a modification to the pressure. Taking the curl gives

$$-2\,\partial\mathbf{U}_{20}/\partial z = \nabla \wedge [(\nabla \wedge \mathbf{B}_{10}) \wedge \mathbf{B}_{10}]. \tag{3.11}$$

From (3.2) and (3.3) we obtain

$$\nabla \wedge \mathbf{B}_{10} = \alpha_{00}\mathbf{B}_{10} - 2C\mathbf{k}, \tag{3.12}$$

where

$$C = -\tfrac{1}{3}B\alpha_{00}J_{\frac{3}{2}}(\alpha_{00}) = \alpha_{00}(12\pi)^{-\frac{1}{2}}\operatorname{sgn}(B) = \pm 0{\cdot}73\ldots.$$

Note that B and C/α_{00} have the same sign. Hence a little manipulation gives

$$\begin{aligned} \nabla \wedge [(\nabla \wedge \mathbf{B}_{10}) \wedge \mathbf{B}_{10}] &= -\nabla \wedge (2C\mathbf{k} \wedge \mathbf{B}_{10}) \\ &= 2C\frac{\partial \mathbf{B}_{10}}{\partial z} = \frac{2C}{\alpha_{00}}\frac{\partial}{\partial z}(\nabla \wedge \mathbf{B}_{10}) \end{aligned}$$

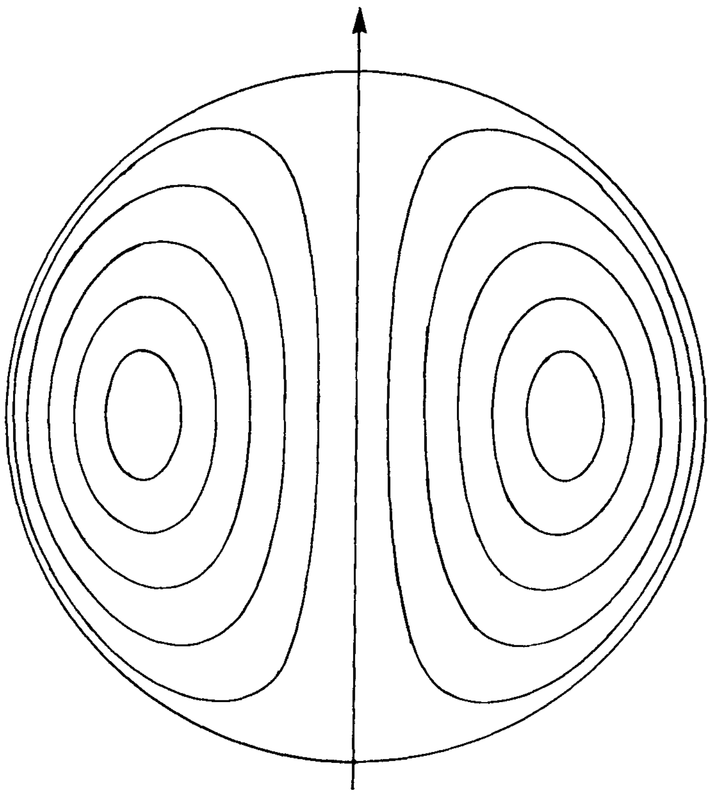

FIGURE 2. Poloidal streamlines of the flow $\mathbf{U}_{20}$, from (3.13), caused by and limiting the amplitude of the magnetic field of figure 1. (Note that the toroidal part of $\mathbf{U}_{20}$, which is driven by the field, is identical in structure with the toroidal part of $\mathbf{B}_{10}$.)

and so from (3.11) we finally obtain

$$\mathbf{U}_{20} = (-C/\alpha_{00})\nabla\wedge\mathbf{B}_{10}+\hat{\mathbf{e}}_{\phi}V_{20}(s), \tag{3.13}$$

and this satisfies $\mathbf{U}_{20}\,.\,\mathbf{n} = 0$ on $r = 1$ as required. We take $V_{20}(s) \equiv 0$ since we are considering the viscous limit, in which Ekman suction dominates any effects of finite amplitude, as discussed in §2.

Figure 2 shows the poloidal part of $\mathbf{U}_{20}$ (the toroidal part of $\mathbf{U}_{20}$ is exactly the same in form as the toroidal part of $\mathbf{B}_{10}$ shown in figure 1). Childress (1969), who conducted a parallel study using dynamos with spatially periodic velocity fields and was the first to obtain this flow, noted that the toroidal flow is 'westward', that is, against the rotation, for *any* of the four possible solutions for $\mathbf{B}_{10}$ found by permuting the signs of B and α_{00}. This same independence of the parity of the solution persists in the case of general f, and it is plausible that the direction of the flow is still 'westward', although this has not been proved.

On $r = 1$, $\mathbf{U}_{20} = (0, U_{2\theta}, 0)$, where

$$U_{2\theta} = -(3/4\pi)^{\frac{1}{2}}C\sin\theta. \tag{3.14}$$

Before setting up the boundary-layer equations, we note that the Taylor integral

$$\int_{C(s)} (\mathbf{B}_{10}.\nabla\mathbf{B}_{10})_\phi dz$$

vanishes identically since $(\mathbf{B}_{10}.\nabla\mathbf{B}_{10})_\phi$ is an odd function of z, and this means that, although we have not attempted to satisfy the Taylor condition, it is automatically satisfied to this order.

3.4. *The boundary-layer equations and the determination of* α_{21}

The first non-trivial boundary-layer equations are

$$\left.\begin{aligned} 2\mathbf{k}\wedge\tilde{\mathbf{U}}_{20} &= \mathbf{n}\,\partial\tilde{p}_{21}/\partial\zeta + \partial^2\tilde{\mathbf{U}}_{20}/\partial\zeta^2, \\ -\partial(\mathbf{n}.\tilde{\mathbf{U}}_{21})/\partial\zeta + \mathbf{n}.\nabla\wedge(\mathbf{n}\wedge\tilde{\mathbf{U}}_{20}) &= 0, \end{aligned}\right\} \tag{3.15}$$

$$\left.\begin{aligned} 0 &= \partial^2\tilde{\mathbf{B}}_{30}/\partial\zeta^2, \\ -(\mathbf{B}_{10}.\mathbf{n})\,\partial\tilde{\mathbf{U}}_{20}/\partial\zeta + \partial^2\tilde{\mathbf{B}}_{31}/\partial\zeta^2 &= 0, \\ -\partial(\mathbf{n}.\tilde{\mathbf{B}}_{31})/\partial\zeta + \mathbf{n}.\nabla\wedge(\mathbf{n}\wedge\tilde{\mathbf{B}}_{30}) &= 0, \end{aligned}\right\} \tag{3.16}$$

with boundary conditions $\mathbf{U}_{20}+\tilde{\mathbf{U}}_{20} = 0$ at $\zeta = 0$ and $\mathbf{B}_{31}+\tilde{\mathbf{B}}_{31}$ continuous across $\zeta = 0$ ($r = 1$) (cf. Greenspan 1968, chap. 2). We see that $\tilde{\mathbf{B}}_{1j} \equiv 0$ since $\mathbf{B}_1$ has no boundary-layer structure and also that the boundary-layer correction to $\mathbf{B}_3$ is $O(E^{\frac{1}{2}})$ owing to the large ohmic dissipation in the boundary-layer region. The solution to (3.15) does not involve the magnetic field and so is that given by Greenspan, viz.

$$\mathbf{n}\wedge\tilde{\mathbf{U}}_{20}+i\tilde{\mathbf{U}}_{20} = -[\mathbf{n}\wedge\mathbf{U}_{20}+i\mathbf{U}_{20}]_{r=1}\exp[-(2i(\mathbf{n}.\mathbf{k}))^{\frac{1}{2}}\zeta], \tag{3.17}$$

where the square root has positive real part and $\mathbf{n}.\mathbf{k} = \cos\theta$.

Since $\mathbf{U}_{21}+\tilde{\mathbf{U}}_{21} = 0$ on $\zeta = 0$, (3.15) and (3.17) give

$$\mathbf{n}.\mathbf{U}_{21}\Big|_{r=1} = -\tfrac{1}{2}\mathbf{n}.\nabla\wedge\{(0,0,U_{2\theta})\,|\cos\theta|^{\frac{1}{2}}\} \tag{3.18}$$

since $\mathbf{U}_{20}|_{r=1} = (0, U_{2\theta}, 0)$ as in (3.14). Details of this analysis may be found in Greenspan (q.v.).

Equations (3.16) give

$$\tilde{\mathbf{B}}_{31}\Big|_{\zeta=0} = -\int_0^\infty (\mathbf{n}.\mathbf{B}_{10})\,\tilde{\mathbf{U}}_{20}\,d\zeta \tag{3.19}$$

and this leads to the boundary condition on b_{31}

$$b_{31}\Big|_{r=1} = -\tilde{b}_{31}\Big|_{\zeta=0} = \frac{1}{2}\frac{\cos\theta}{|\cos\theta|^{\frac{3}{2}}}U_{2\theta}(\mathbf{n}.\mathbf{B}_{10})\Big|_{r=1} \equiv b(\theta) \tag{3.20}$$

since $b_{31}+\tilde{b}_{31} = 0$ on $r = 1$. There are similar conditions on the other components of $\mathbf{B}_{31}$, but they may be easily accommodated since the poloidal part of $\mathbf{B}_{31}$ is only determined to within a potential field and this can be varied to satisfy the new boundary conditions without α having to be changed.

The equations for $\mathbf{U}_{21}$ and $\mathbf{B}_{31}$ are

$$\left.\begin{aligned} 2\mathbf{k}\wedge\mathbf{U}_{21} &= -\nabla p_{21}, \quad \nabla.\mathbf{U}_{21} = 0, \\ 0 &= c_{31}+\alpha_{21}b_{10}+\alpha_{00}b_{31}+D^2a_{31}, \\ 0 &= d_{31}-\alpha_{21}D^2a_{10}-\alpha_{00}D^2a_{31}+D^2b_{31} \end{aligned}\right\} \tag{3.21}$$

since $\mathbf{B}_{11} = 0$; the boundary conditions on these fields are given by (3.18) and (3.20). Then

$$\mathbf{U}_{21} = W(s)\,\mathbf{k} \tag{3.22}$$

is the solution for the velocity field, where $W(s)$ is determined by (3.18). We see, however, that since $\mathbf{U}_{21}$ and $\mathbf{U}_{20}$ are even functions of z we may make use of the parity arguments of §3.2 to show that α_{21} is not affected by this flow, just as α_{20} was not affected by $\mathbf{U}_{20}$. Hence the only contribution to α_{21} must come from the effect of the boundary condition on b_{31}. If we ignore $\mathbf{U}_{21}$ and the fields it drives, we have

$$0 = 2\alpha_{21}\alpha_{00}b_{10}+L_0(b_{31}) \tag{3.23}$$

from (3.2). Hence

$$0 = 2\alpha_{21}\alpha_{00}+\{b_{10}L_0(b_{31})\}$$

and the quantity in brackets can be written as

$$\{b_{31}L_0(b_{10})\}-\int_{\partial V} b_{31}\frac{\partial b_1}{\partial r}dS = \int_{\partial V} b(\theta)\,(\nabla\wedge\mathbf{B})_\theta dS, \tag{3.24}$$

where ∂V is the surface $|\mathbf{r}| = 1$. The divergence theorem and the equation for b_{10} from (3.2) have been used. Equations (3.23), (3.24), (3.13) and (3.14) then imply that

$$\alpha_{21} = \frac{1}{2C}\int_{\partial V} U_{2\theta}b(\theta)\,dS \tag{3.25}$$

and from (3.14), (3.20) and (3.25) with (3.3) then lead to the very simple expression

$$\alpha_{21} = 4C^2/7\alpha_0 = 6{\cdot}79\ldots\ . \tag{3.26}$$

3.5. *Discussion*

We have shown in this section that the first correction to α_{00} is $O(\epsilon^2E^{\frac{1}{2}})$ and has the same sign as α_{00}. We would expect this since an increase in $|\alpha|$ implies an increase in the energy fed in from the small scales, and this energy is dissipated by viscosity.

It is interesting to note that the expression (3.26) is identical to the value of the integral

$$\frac{1}{2\alpha_{00}}E^{-\frac{1}{2}}\left[E^{\frac{1}{2}}\times 2\pi\int_0^\pi\int_0^\infty d\theta\, d\zeta\frac{\partial\tilde{\mathbf{U}}_{20}}{\partial\zeta}.\frac{\partial\tilde{\mathbf{U}}_{20}}{\partial\zeta}\right], \tag{3.27}$$

as may be verified from (3.17), where the term in brackets is $\Phi_{\nu4}$, the viscous dissipation due to the $\mathbf{U}_{20}$ field, to leading order in $E^{\frac{1}{2}}$. Hence the relationship between α and ϵ^2 takes the form

$$\alpha = \alpha_{00}+(\epsilon^2/2\alpha_{00})\,\Phi_{\nu4}+\ldots\ . \tag{3.28}$$

The equality of the values of (3.27) and (3.25) is essential at this order since there is now a precise balance between the extra energy fed in from the small scales and the energy dissipated by viscosity at finite amplitude. There is no additional ohmic dissipation at this level because of the field parities, and we have to go to $O(\epsilon^5)$ to find the first effects of field distortion.

4. Solutions in the inviscid limit ($E^{\frac{1}{4}} \ll \epsilon \ll 1$)

4.1. *The effect of Taylor's condition*

We now suppose that as long as $\epsilon^2 \gg E^{\frac{1}{4}}$ we may approximate the solution to the full equations by setting $E = 0$ and ignoring the effect of boundary layers. We shall show in the next section that this approximation is valid as far as we need to use it. The fact that restrictions must be placed on the relative magnitude of ϵ and E is enough to show that the solution for $E = 0$ is not approached uniformly for all ϵ as E becomes small. However, we shall find that the regions of ϵ, E space where such a limit is valid are the regions we would expect to be important in the earth's core.

In the absence of boundary layers, the equation of motion (2.4) is subject to the Taylor consistency condition (1.4). For axisymmetric fields this can be written in the form

$$T(s) \equiv \int_{C(s)} (\mathbf{B}.\nabla\mathbf{B})_\phi dz \equiv 0, \tag{4.1}$$

since the ϕ integration is trivial. The meaning of (4.1) is that there are some zonal Lorentz torques which cannot be balanced by Coriolis forces, and so certain restrictions must be placed on these torques so that they do not induce large fluid accelerations, violating the assumptions on which the derivation of (2.4) was based. It would seem that this extra constraint on $\mathbf{B}$ would overdetermine the problem, but this is not so since we may write the solution of (2.4) as

$$\mathbf{U} = \mathbf{U}_{\text{driven}} + \hat{\mathbf{e}}_\phi V(s) \tag{4.2}$$

[cf. (3.13)], where $V(s)$ is a 'free' zonal flow which does not depend on the structure of the Lorentz force, and so is free to be determined by the constraint (4.1). Thus, in principle, the velocity field (and hence the magnetic field) can be determined as that which leads to the satisfaction of (4.1). Unfortunately, we do not know of any algorithm for finding the solutions in a constructive manner, and so cannot prove either the existence or uniqueness of such solutions. We shall, however, exhibit solutions in certain special cases.

We remark that from (4.1) and (4.2) it may easily be seen that

$$\{\hat{\mathbf{e}}_\phi V(s).(\mathbf{B}.\nabla\mathbf{B})\} = 0, \tag{4.3}$$

in other words, the 'free' velocity field $V(s)$ makes no contribution to the energy balance, as we would expect since $V(s)$ is not driven by the field. $V(s)$ can be regarded as satisfying a consistency condition equivalent to (4.1).

One question remaining is that of how the fields attain a state in which (4.1) is satisfied. If we suppose that a magnetic field that does not satisfy (4.1) is 'switched on' in the field, the fluid will execute high frequency torsional oscillations on the time scale $E_M^{-\frac{1}{2}}\Omega^{-1}$ about a state in which (4.1) is satisfied; the term $E_M \partial\mathbf{U}/\partial t$ becomes important on this time scale if $|\mathbf{U}| = O(E_M^{-\frac{1}{2}})$. The time scale of these motions is of the order of the Alfvén time scale, and the motions resemble torsional Alfvén waves (Braginskii 1967; Roberts & Soward 1972). The waves will die away owing to ohmic losses in the distorted fields, and owing to Ekman suction if some viscosity is present. If we are considering only steady solutions (and we shall see later in this section that the model we describe is far more likely to have steady solutions than oscillatory ones), then the ohmic loss mechanism is enough to ensure that a steady state exists. If we were to consider oscillating solutions with time scales of the order of the ohmic decay time, it would be necessary to invoke Ekman suction as the damping mechanism, since this damps the oscillations over the spin-up time $E^{-\frac{1}{2}}\Omega^{-1}$, which for most estimates of E is shorter than the ohmic decay time. For these reasons we shall therefore suppose henceforth that (4.1) is satisfied for all time.

We are now in a position to solve the finite amplitude problem for $f = 1$, and this is done in the next subsection. The problem for general f, however, has many interesting features not found in the case considered and we discuss these in the final subsection.

4.2. *The eigenvalue problem for $f = 1$*

From (2.9) and (2.11) we obtain

$$\left.\begin{aligned} q_0 \partial a_1/\partial\tau &= \alpha_0 b_1 + D^2 a_1, \\ q_0 \partial b_1/\partial\tau &= -G_0(s)\,\partial a_1/\partial z - \alpha_0 D^2 a_1 + D^2 b_1, \end{aligned}\right\} \tag{4.4}$$

where $G_0(s) \equiv s d[s^{-1}V_0(s)]/ds$ and $\mathbf{U}_0 = \hat{\mathbf{e}}_\phi V_0(s)$, together with the Taylor condition

$$T_2(s) \equiv \int_{C(s)} (\mathbf{B}_1 . \nabla \mathbf{B}_1)_\phi \, dz \equiv 0. \tag{4.5}$$

Equations (4.4) and (4.5), together with the boundary conditions on a_1 and b_1, which are the same as in §3, constitute an eigenvalue problem in which not only q_0 and α_0 but also $V_0(s)$ must be determined as part of the eigensolution. In the case of general f, this is next to impossible to solve, but in the case $f = 1$ it is trivial. If we assume that the lowest eigenvalue occurs for $q_0 = 0$, as seems likely, we have the following.

(i) A consistent solution exists for $G_0(s) \equiv 0$; for in the absence of $G_0(s)$, the solution is identical to that in §3, and we have shown by parity arguments that (4.5) is satisfied for this solution.

(ii) If $G_0(s) \not\equiv 0$, then parity considerations show that the solution could have no definite parity about $z = 0$, since if it did, the term in $G_0(s)$ would be of different parity to all the others.

(iii) Any eigenvalue for a steady solution with $G_0(s) \not\equiv 0$ would be greater than the α_0 defined by (3.3); for (4.1) may be integrated by parts to obtain the equivalent form (Childress 1969)

$$\int_{C(s)} b \frac{\partial a}{\partial z} dz \equiv 0. \tag{4.6}$$

Hence (4.5) can be written as

$$\int_{C(s)} b_1 \frac{\partial a_1}{\partial z} dz \equiv 0. \tag{4.7}$$

If this is used and we repeat the analysis of appendix A then it is easy to show that the real part of equation (A7) remains unaltered, and so

$$\alpha_0^2 = \left\{|\nabla b_1|^2 + \frac{|b_1|^2}{s^2}\right\} \Big/ \{|b_1|^2\}. \tag{4.8}$$

It is clear from this equation and (3.4) that the minimum value of α_0^2 for a steady solution must be obtained from the solution to (3.2). It also seems most plausible that $q_0 = 0$ for all possible solutions, though we have not proved this.

So in this case the eigenvalue problem is identical with that in § 3, and so we may take over all the results proved there up to and including the determination of $\alpha_{20} = \alpha_2$. (We shall drop the second suffixes in this section, since we have neglected the effect of viscosity.)

4.3. *The determination of* α_4

Since $\alpha_2 = 0$ we must go to α_4 to find the closure of the equations. The analogues of (3.6) are

$$\left.\begin{aligned} 2\mathbf{k} \wedge \mathbf{U}_4 &= -\nabla p_4 + \mathbf{B}_1 . \nabla \mathbf{B}_3 + \mathbf{B}_3 . \nabla \mathbf{B}_1, \\ 0 &= c_5 + \alpha_4 b_1 + \alpha_0 b_5 + D^2 a_5, \\ 0 &= d_5 - \alpha_4 D^2 a_1 - \alpha_0 D^2 a_5 + D^2 b_5, \end{aligned}\right\} \tag{4.9}$$

and we most now satisfy the Taylor constraint

$$T_4(s) \equiv \int_{C(s)} (\mathbf{B}_1 . \nabla \mathbf{B}_3 + \mathbf{B}_3 . \nabla \mathbf{B}_1)_\phi \, dz \equiv 0. \tag{4.10}$$

We note that, since b_3 has opposite parity to b_1 from (3.7) and (3.8), this integral is not trivial. It is used to determine the part of $\mathbf{U}_2$ which is left undetermined by the equation of motion as discussed earlier in this section; the driven part of $\mathbf{U}_{20}$ is the same as that in § 3. (Note that (3.9) remains true whatever $V_2(s)$ is, and therefore the result $\alpha_2 = 0$ still stands.) The equation analogous to (3.7) is

$$0 = \alpha_0 c_5 + d_5 + 2\alpha_4 \alpha_0 b_1 + L_0(b_5) \tag{4.11}$$

and so the solvability condition (2.15) becomes

$$\alpha_4 = (-2\alpha_0)^{-1} \{b_1(\alpha_0 c_5 + d_5)\}, \tag{4.12}$$

the appropriate part of the identity (3.9) becomes

$$\{b_1 d_5 - D^2 a_1 c_5\} + \{b_3 d_3 - D^2 a_3 c_3\} = 0 \tag{4.13}$$

and after some manipulation, using (3.6) and (3.2), we find that

$$\begin{aligned} \{b_1(\alpha_0 c_5 + d_5)\} &= -\{b_3(\alpha_0 c_3 + d_3)\} - \{c_3^2\} \\ &= -[\{c_3^2\} - \alpha_0^2 \{b_3^2\} + \{|\nabla b_3|^2 + |b_3|^2/s^2\}]. \end{aligned}$$

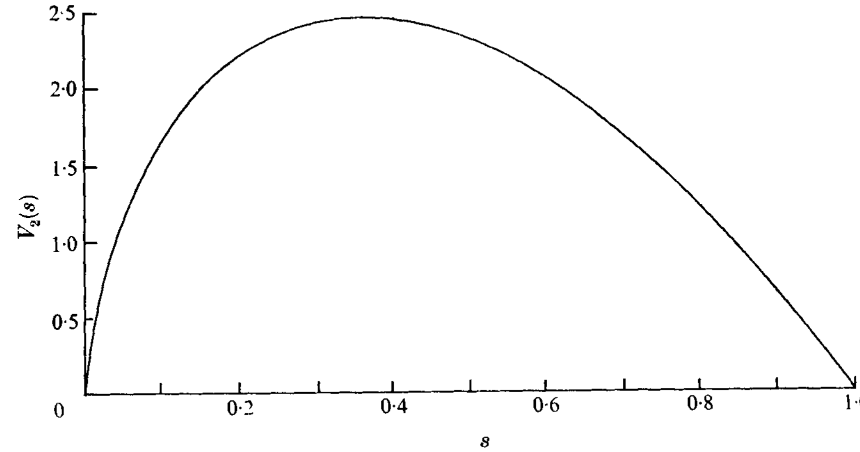

FIGURE 3. The 'free' zonal flow $V_2(s)$ as a function of s; the undetermined solid-body rotation in $V_2(s)$ has been chosen to vanish at $s = 1$, in accordance with the results of §5.

So we finally obtain

$$\alpha_4 = (2\alpha_0)^{-1}[\{c_3^2\} - \alpha_0^2\{b_3^2\} + \{|\nabla b_3|^2 + |b_3|^2/s^2\}], \tag{4.14}$$

and this expression is positive definite from (3.4). Note that we do not have to find $\mathbf{U}_4$ to determine α_4.

Here, finally, is the result anticipated in the introduction: a brake on the growth of the field due to ohmic dissipation, since the terms in b_3 are representative of the added loss due to distortion of the b_1 field, which is the most efficient configuration for transforming the energy of the small scales to energy of the large scales through the term in α. The $\{c_3^2\}$ term is representative of the 'back e.m.f.'s' which arise in response to the growth of the velocity field.

We should remark that the b_3 of this section is not the b_{30} of §3, since the $\mathbf{U}_2$ field is modified by the need to satisfy (4.10). We have found no way to obtain $V_2(s)$, the free part of $\mathbf{U}_2$, analytically, the main difficulty being that the constraint (4.10) is over cylindrical surfaces, whereas because of the boundary conditions the problem is only separable in spherical polars. However, it has proved possible, by expanding in eigenfunctions of the problem for b_1, to find the form of $V_2(s)$ by a fairly straightforward numerical iteration procedure, the details of which will appear elsewhere. We give the resulting form for $V_2(s)$ in figure 3. Note that the flow is 'eastwards', opposite to the driven part of the toroidal flow. This is to be expected since the velocity field will draw out the toroidal part of the magnetic field, and we would perhaps expect that (4.10) can only be satisfied if the 'pulling out' is not all of one sign. Because the ϕ component of the driven part of $\mathbf{U}_2$ is zero at $r = 1$, see (3.14), this means that at the surface of the core the flow is eastwards, but that in the interior the flow is westwards near the equator and eastwards near the poles.

Although there does not seem to be much point in calculating the exact value of α_4, since its analytical form is much more revealing, it is certainly positive, and $O(1)$. We have therefore justified the scaling and ordering of §2 by showing explicitly that, for ϵ^2 sufficiently large compared with $E^{\frac{1}{4}}$, the energy balance is determined primarily by ohmic rather than viscous dissipation. Had all the α_{j0} turned out to be zero, we would have had to revise our scaling to make viscous dissipation the important mechanism.

4.4. *Discussion*

We conclude this section with some remarks on the structure of the problem in the inviscid limit for general f. If we suppose that f is either odd or even, which is plausible bearing in mind the likely symmetries of the earth's core about the equator and the probable importance of $\mathbf{\Omega}$ in the production of the α-effect, then marked differences arise between the two cases.

Case 1. *f even.* In this case it is easy to show from the equations for the eigenvalue problem [cf. (2.9) and (4.4)]

$$q_0 \partial a_1/\partial \tau = \alpha_0 f b_1 + D^2 a_1, \tag{4.15a}$$

$$q_0 \partial b_1/\partial \tau = -G_0(s)\, \partial a_1/\partial z + \alpha_0\, \hat{\mathbf{e}}_\phi \,.\, \nabla \wedge [f \nabla \wedge (a_1 \hat{\mathbf{e}}_\phi)] + D^2 b_1 \tag{4.15b}$$

and parity arguments similar to those used in §§3 and 4.2 that, if a solution is sought for which a_1, say, has definite parity, then b_1 has the same parity and (4.5) is automatically satisfied if and only if $G_0(s) \equiv 0$. Since it is most likely that the lowest eigenvalue α_0 will be obtained for a solution having definite parity, we have the situation that for even f the eigenvalue problems in the viscous and inviscid limits are identical. Further application of these arguments shows that since $b_1^{\dagger}$ will have the same parity as b_1, and c_3 and d_3 will have opposite parity, α_2 will vanish to leading order in $E^{\frac{1}{4}}$, just as in the $f = 1$ case. An extension of this argument shows that $\alpha_6 = \alpha_{10} = \ldots = 0$ and that the only non-trivial Taylor integrals occur at orders ϵ^4, ϵ^8, etc., so that the 'free' parts of the $\mathbf{U}_j$ have to be determined only at $j = 2, 6$, etc. Furthermore, the fields at successive orders in ϵ^2 alternate in parity, so that the finite amplitude field has no symmetries about the equator.

Case 2. *f odd.* In this case a_1 and b_1 have opposite parities [from (4.14)] and $G_0(s) \not\equiv 0$ in general, since (4.5) is not trivially satisfied. Hence the eigenvalue problems in the two limits are very different. In this case α_{20} will not be zero in general, and the free part of $\mathbf{U}_j$ has to be determined for all j, since none of the Taylor integrals are trivial. Further, all the b_j have the same parity as b_1, etc. and so the finite amplitude field will have the same symmetries as the infinitesimal amplitude field.

These differences are very striking, and should help to provide evidence for or against even or odd f's in the earth or sun. It is far more likely that f is odd, and Roberts (1972), in his comprehensive numerical study of α-effect dynamos, always uses an f that is odd, but does not require his solutions to satisfy (4.5). His results, we think, need reappraisal in the light of the foregoing discussion. It seems that the eigenvalues he obtains are too low, since the additional constraint (4.5) would tend to increase the ohmic dissipation since it would add structure to the field, and so a higher energy input (via the α term) would be required. A numerical study of the effects of (4.5) on Roberts' results is in progress.

Roberts found that all his solutions for α^2-dynamos (of the type studied here, in which no large-scale motions are prescribed) had steady eigensolutions, and we expect the same to be true even for non-zero $G_0(s)$, since we have already shown that the term in $G_0(s)$ does not contribute to the energy transfer (4.3), and so cannot replace the term in α_0 in (4.15*b*), as a source of energy. So the system is still basically an α^2-dynamo, and we expect steady eigensolutions, although no general proof of marginal stability exists as yet.

We have now elucidated the finite amplitude effects on the problem in the two extremes $\epsilon^2 \to 0$ and $\epsilon^2 \gg E^{\frac{1}{4}}$. We now turn to a brief exposition of the situation between these two extremes.

5. The transition region

5.1. *Preamble*

We have seen that in §3 the free part of the velocity field $V(s)$ was taken to be zero owing to Ekman suction, and in §4 the Taylor condition $T(s) \equiv 0$ was assumed to be satisfied. Since in general both these requirements are impossible to fulfil, we would expect that the situation in intermediate ranges of ϵ, E space is characterized by a balance between Ekman suction and the effect of the magnetic field through the Taylor condition, as the system attempts to satisfy both at once. In this section we attempt to describe this balance for general f in the two cases $E^{\frac{1}{4}} \gg \epsilon^2 \gg E^{\frac{1}{2}}$ and $E^{\frac{1}{2}} \gg \epsilon^2$ where the solution is 'near' the inviscid and viscous limits respectively. The case $f = 1$ is then described by reference to the general treatment. Following the discussion in §4, we shall for simplicity consider only steady solutions.

5.2. *Case* 1. $\epsilon^2 \ll E^{\frac{1}{2}} \ll 1$

If we expand all fields in powers of ϵ^2 and $E^{\frac{1}{2}}$ as in §3, we obtain in the interior

$$2\mathbf{k} \wedge \mathbf{U}_{00} = -\nabla p_{00}, \tag{5.1}$$

$$2\mathbf{k} \wedge \mathbf{U}_{20} = -\nabla p_{20} + \mathbf{B}_{10} \cdot \nabla \mathbf{B}_{10} + \mathbf{F}, \tag{5.2}$$

$$2\mathbf{k} \wedge \mathbf{U}_{01} = -\nabla p_{01} - \mathbf{F}', \tag{5.3}$$

$$\nabla \cdot \mathbf{U}_{00} = \nabla \cdot \mathbf{U}_{20} = \nabla \cdot \mathbf{U}_{01} = 0 \tag{5.4}$$

and in the boundary layers

$$\left.\begin{aligned} 2\mathbf{k} \wedge \tilde{\mathbf{U}}_{00} &= \mathbf{n}\,\partial \tilde{p}_{01}/\partial\zeta + \partial^2 \tilde{\mathbf{U}}_{00}/\partial\zeta^2, \\ -\partial(\mathbf{n} \cdot \tilde{\mathbf{U}}_{01})/\partial\zeta + \mathbf{n} \cdot \nabla &\wedge (\mathbf{n} \wedge \tilde{\mathbf{U}}_{00}) = 0. \end{aligned}\right\} \tag{5.5}$$

$\mathbf{U}_{00}$ and $\mathbf{U}_{20}$ obey inviscid boundary conditions (cf. §3). $\mathbf{F}$ and $\mathbf{F}'$ are undefined for the present. The boundary condition on $\mathbf{U}_{01}$ is found from the boundary-layer equations to be

$$\mathbf{U}_{01} \cdot \mathbf{n}\Big|_{r=1} = -\tfrac{1}{2}\mathbf{n} \cdot \nabla \wedge \left[\left(\left(0, 0, \frac{|\cos\theta|}{\cos\theta} V_{00}(s)\right) |\cos\theta|^{\frac{1}{2}}\right)\Big|_{r=1}\right], \tag{5.6}$$

where $\mathbf{U}_{00} = \hat{\mathbf{e}}_\phi V_{00}(s)$ as in (2.11). The analysis required to obtain this is identical with that for (3.17) and (3.18) and can be found in Greenspan (1968, q.v.). Note that the right-hand side of (5.6) is an even function of z.

Two difficulties now become apparent. If we supposed that $\mathbf{F} = \mathbf{F}' = 0$ then (i) the Taylor condition in the equation for $\mathbf{U}_{20}$ would not be satisfied in general and (ii) no solutions to (5.3) would exist. For the general solution to (5.3) with $\mathbf{F}' = 0$ is

$$\mathbf{U}_{01} = W_{01}(s)\,\mathbf{k} + V_{01}(s)\,\hat{\mathbf{e}}_\phi, \tag{5.7}$$

and this would give $\mathbf{U}_{01}.\mathbf{n}\,|_{r=1}$ as an odd function of z, which is incompatible with (5.6).

In order to solve both these problems at once, let us choose $\mathbf{F}$ so that the Taylor condition for (5.2) is satisfied, i.e. choose

$$\mathbf{F} = F(s)\,\hat{\mathbf{e}}_\phi, \quad F(s) = -\frac{1}{2(1-s^2)^{\frac{1}{2}}}\int_{C(s)} (\mathbf{B}_{10}.\nabla\mathbf{B}_{10})_\phi\,dz. \tag{5.8}$$

We must then take $\mathbf{F}' = \eta\mathbf{F}$, where $\eta = \epsilon^2 E^{-\frac{1}{2}}$, as can be seen by consideration of the relative orders of magnitude of (5.2) and (5.3). Note that $\eta \ll 1$. The general solution to (5.3) and (5.4) compatible with (5.6) is, from Greenspan (1968, p. 47),

$$\mathbf{U}_{01} = +\eta\nabla\wedge[\tfrac{1}{2}zF(s)\,\hat{\mathbf{e}}_\phi] + V_{01}(s)\,\hat{\mathbf{e}}_\phi \tag{5.9}$$

and substituting this in (5.6) leads to the relation

$$V_{00}(s) = \tfrac{1}{2}\eta(1-s^2)^{\frac{1}{4}}\int_{C(s)} (\mathbf{B}_{10}.\nabla\mathbf{B}_{10})_\phi\,dz. \tag{5.10}$$

This is the fundamental relationship between the Lorentz forces and Ekman suction that is required.† It is now clear that at zeroth order in η we have the viscous solution as before, and that all the quantities should be expanded in powers of η as well as ϵ^2 and $E^{\frac{1}{2}}$ (the expansion in ϵ^2 becomes redundant except for the odd ϵ's in the magnetic equations). In the general-f case α can be written as

$$\alpha = \alpha_{00} + \eta\alpha' + E^{\frac{1}{2}}\alpha_{01} + \eta E^{\frac{1}{2}}\alpha_{20} + \ldots, \tag{5.11}$$

and clearly the leading correction to α is $\eta\alpha'$, since α_{01} must be zero (without *any* finite amplitude effects there can be no velocity field, and so no viscous effects).

5.3. *Case* 2. $E^{\frac{1}{2}} \ll \epsilon^2 \ll 1$

In this region we would expect solutions 'near to' the inviscid solutions of § 4, and so would expect the Taylor condition to be nearly satisfied rather than $\mathbf{U}_{00}$ to be nearly zero. We may make use of all the equations in § 5.2, with the difference that the small parameter is now $\eta^{-1} = \epsilon^{-2}E^{\frac{1}{2}}$ instead of η, and that boundary-layer effects provide a small correction to finite amplitude effects instead of the other way round. All the analysis goes through just as before and we obtain

$$\int_{C(s)} (\mathbf{B}_{10}.\nabla\mathbf{B}_{10})_\phi\,dz = 2\eta^{-1}(1-s^2)^{-\frac{1}{4}}V_{00}(s), \tag{5.12}$$

† This expression should be compared with the one derived by Tough & Roberts (1968) in connexion with the magnetohydrodynamic Braginskii dynamo.

which is now an expression for small deviations from zero of the Taylor condition. To first order in η^{-1}, we may take $V_{00}(s)$ as the value obtained if viscosity is negligible. (Incidentally, it is clear from (5.12) that $V_{00}(s)$ must tend to zero as $s \to 1$.) We are able to ensure that this is the case since $V_{00}(s)$ is only determined up to a solid-body rotation by the eigenvalue equations (2.9). So in this region it is appropriate to introduce η^{-1} as another expansion parameter and the leading terms in the expansion for α become

$$\alpha = \alpha_{1\mathrm{nv}} + \eta^{-1}\alpha'' + \epsilon^2\alpha_{20} + \epsilon^2\eta^{-1}\alpha_{01} + \ldots, \tag{5.13}$$

and clearly the term in η^{-1} will be the most important if and only if $\eta^{-1} \gg \epsilon^2$, i.e. $\epsilon^2 \ll E^{\frac{1}{4}}$. If $\epsilon^2 \gg E^{\frac{1}{4}}$, we have the inviscid limit of §§2 and 4, in which viscosity can be neglected. It is important to note that the $\alpha_{1\mathrm{nv}}$ of (5.13) is the *inviscid* eigenvalue and is therefore not the same as the α_{10} in (5.11). Since the viscous eigenvalue is likely to be much less than the inviscid one (since the inviscid eigensolution will have more structure; this was discussed in §4), we expect that α' and α'' are, respectively, positive and negative.

5.4. *Solution for $f = 1$*

If $f = 1$ (or, indeed, if f is any even function of z), then the eigenvalue problem is unaffected by the effects of finite amplitude or viscosity, and so there will be no $O(\eta)$ or $O(\eta^{-1})$ corrections to the solutions. The appropriate expansion for α will therefore be

$$\alpha = \alpha_{00} + \epsilon^2\eta\alpha' + \epsilon^2E^{\frac{1}{2}}\alpha_{21} + \ldots \tag{5.14}$$

for $\eta \ll 1$, and

$$\alpha = \alpha_{00} + \epsilon^2\eta^{-1}\alpha'' + \epsilon^4\alpha_{40} + \ldots \tag{5.15}$$

for $\eta^{-1} \ll 1$.

However, it is an easy matter to show that α' and α'' vanish from parity arguments and so the conditions for η and η^{-1} to be important are, respectively,

$$\eta^2 \gg E^{\frac{1}{2}}, \quad \text{i.e.} \quad \epsilon^2 \gg E^{\frac{3}{4}} \quad (\text{and } \epsilon^2 \ll E^{\frac{1}{2}}), \tag{5.16}$$

and

$$\eta^{-2} \gg \epsilon^2, \quad \text{i.e.} \quad \epsilon^2 \ll E^{\frac{1}{8}} \quad (\text{and } \epsilon^2 \gg E^{\frac{1}{2}}), \tag{5.17}$$

which shows that the conditions given in §2 for the viscous and inviscid limits to hold are sufficient, but not necessary, in the particular case f even. However, the discussion given in the previous two subsections shows that the conditions are necessary and sufficient in the general case.

5.5. *Discussion*

We have now shown how the conditions stated in §2 arise and have provided a means for bridging the gap between the two limits of §§3 and 4. Quantitative results are hard to obtain because, as in §4, there is a cylindrical constraint to be satisfied with spherical boundary conditions. However, we are now in a position to show how α depends on ϵ, and this is shown in figure 4 (general-f case) and figure 5 (constant-f case). Figure 6 shows the different regions of ϵ, E space that have been described.

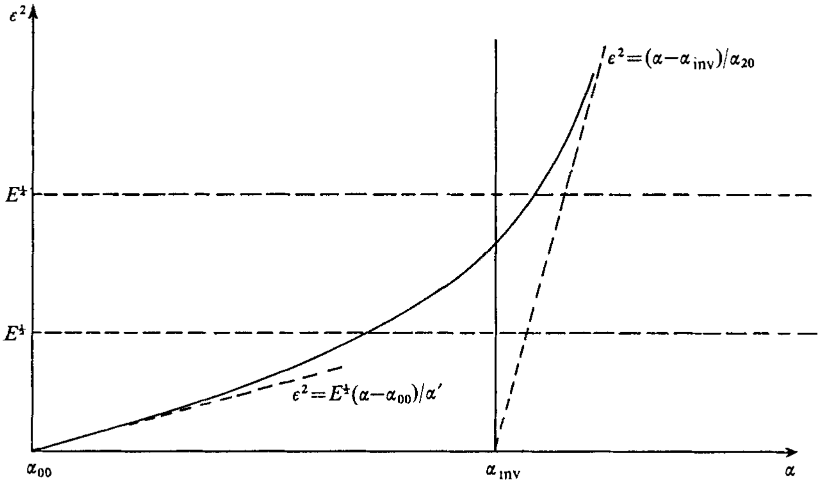

FIGURE 4. The relationship between the magnetic energy ($\propto \epsilon^2$) and $\Delta\alpha \equiv \alpha - \alpha_{00}$ for general f. Here α_{00} is taken as the 'viscous' eigenvalue, with α_{inv} as the 'inviscid' eigenvalue.

We would expect that the earth's field is in the 'inviscid' regime of § 4, for observations of fields within the earth's core lead to the conclusion that $\epsilon = O(10^{-1})$ at least, whereas $E = O(10^{-14})$, and so $\epsilon^2 \gg E^{\frac{1}{4}}$. Hence we have shown that the primary factor inhibiting the growth of the magnetic field is ohmic dissipation, as anticipated in the scaling.

6. Conclusion

In previous sections, we have shown that a finite amplitude equilibration of a field driven by the α-effect can occur owing to large-scale velocity fields in the domain. The principal dissipation mechanism is found to be ohmic loss except at very small field amplitudes where viscous dissipation is also important. In this respect, the system resembles an unloaded electric motor. The α^2-dynamo that we have chosen as a basis for the analysis has several shortcomings as a model for the generation of the earth's and sun's magnetic fields; the lack of oscillatory solutions, in particular, makes it irrelevant to the sun. However, we anticipate that if a similar analysis were applied to the more appropriate $\alpha\omega$-dynamo (in which a prescribed differential rotation, instead of the α-effect, causes the production of toroidal from poloidal field) then the same type of equilibration would occur.

Two remaining questions concern the neglect of the $\nabla.\mathbf{R}$ and $\mathbf{F}$ terms in (1.3), and the suppression of the dependence of αf on the large-scale fields. We have left out the term in $\mathbf{F}$ to simplify the model; we could equally have included it and considered the resulting $\alpha\omega$-dynamo, with essentially the same conclusions. Neglect of $\nabla.\mathbf{R}$ can only be justified by reference to a particular small-scale process. However, using the estimates in Moffatt (1972), we find that

$$|\nabla.\mathbf{R}| \simeq |u_0|^2/L \simeq 0{\cdot}005\,\Omega\lambda/L,$$

where the small-scale motions have speeds $\simeq |u_0|$, so that $\nabla.\mathbf{R}$ can certainly be neglected in this case.

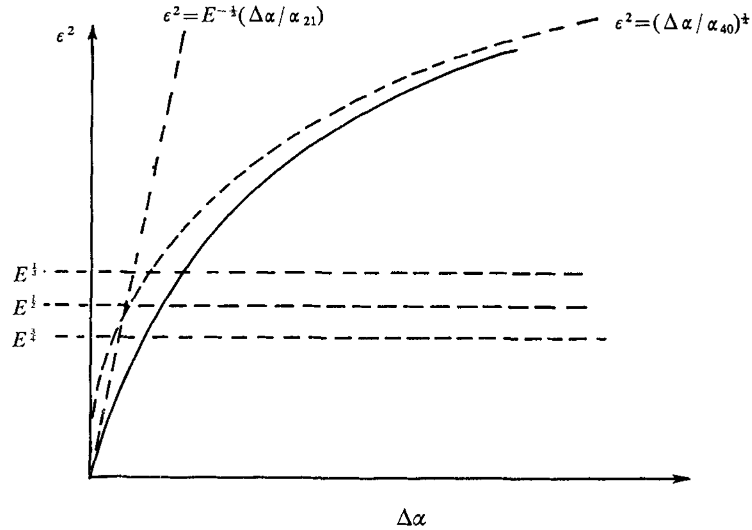

FIGURE 5. The relationship between the magnetic energy ($\propto \epsilon^2$) and $\Delta\alpha \equiv \alpha - \alpha_{00}$ for $f = 1$ (and, by extension, for general *even f*). Note that the 'viscous' and 'inviscid' eigenvalues are the same in this case.

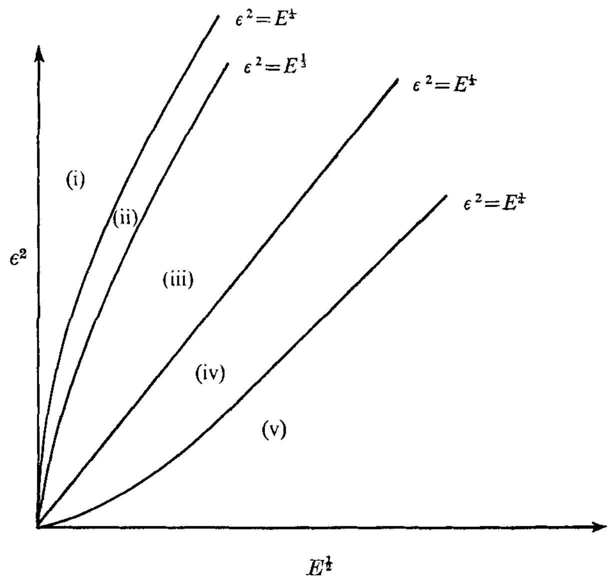

FIGURE 6. The various regions in ϵ, E space considered in §§2–5. (i) $1 \gg \epsilon^2 \gg E^{\frac{1}{4}}$: inviscid region. (ii) $E^{\frac{1}{4}} \gg \epsilon^2 \gg E^{\frac{1}{3}}$: transition region for general f; inviscid region for even f. (iii) $E^{\frac{1}{3}} \gg \epsilon^2 \gg E^{\frac{3}{4}}$: transition region. (iv) $1 \gg E^{\frac{3}{4}} \gg \epsilon^2$: transition region for general f; viscous region for even f. Note that ϵ^2 must tend to zero in the viscous limit for general f.

The same considerations apply to the investigation of the relative importance of the equilibration mechanism found in this paper and that of αf reduction due to changes in the small-scale fields as **B** and **U** increase.† If we again consider Moffatt's (1972) process, we find that the relative importance of the two mechanisms depends crucially on the spectrum of the small-scale velocity field; in general, it seems that both mechanisms can be important, depending on the region of parameter space being considered. However, the observed magnitude of the geomagnetic field (Roberts & Soward 1972) strongly suggests the magnetostrophic equilibration described in this paper.

One of the most intriguing aspects of this study is the novel eigenvalue problem outlined in §4. A necessary part of the eigensolution is the determination of a zonal flow which will prevent the growth of any magnetic fields causing zonal torques. This 'eigenflow' is independent of ϵ and hence of $\Delta\alpha$. Unfortunately, it is difficult to obtain analytic solutions for non-trivial f's, although in an ingeneous approach to the problems raised in §4 (Greenspan 1974), solutions have been found for f's which are non-zero only in thin shells. A study is now in progress which will include the Taylor condition within a numerical method for solving the eigenvalue problem for general f, and this will hopefully show to what extent the results of Roberts (1972) and others need reappraising and in particular how their criteria for time-independent solutions for $\alpha\omega$-dynamos are affected.

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Appendix A. Proof of the principle of exchange of stabilities for (3.1)

If we take (3.1) and its complex conjugate

$$\left.\begin{aligned} 0 &= iq_0 a_1^* + \alpha_0 b_1^* + D^2 a_1^*, \\ 0 &= iq_0 b_1^* - \alpha_0 D^2 a_1^* + D^2 b_1^*, \end{aligned}\right\} \quad \text{for} \quad |\mathbf{r}| \leqslant 1 \tag{A 1}$$

(q_0 is real for neutrally stable solutions), then we may easily obtain the three integral relations

$$\left.\begin{aligned} -iq_0\{|a_1|^2\} &= \alpha_0\{a_1 b_1^*\} + \{a_1 D^2 a_1^*\}, \\ iq_0\{|b_1|^2\} &= -\alpha_0\{b_1^* D^2 a_1\} + \{b_1^* D^2 b_1\}, \\ iq_0\{a_1 b_1^*\} &= \alpha_0\{|b_1|^2\} + \{b_1^* D^2 a_1\}. \end{aligned}\right\} \tag{A 2}$$

Now from the equation for a_1 for $|\mathbf{r}| \geqslant 1$, $D^2 a_1 = 0$, we can show that for $|\mathbf{r}| \geqslant 1$

$$a_1 = \sum_1^\infty A_N r^{-(N+1)} P_N^1(\cos\theta)$$

and so

$$\left.\frac{\partial a_1}{\partial r} = -\sum_1^\infty (N+1) A_N r^{-(N+2)} P_N^1(\cos\theta).\right\} \tag{A 3}$$

† Rüdiger (1973) has recently investigated the equilibration due to this mechanism.

This implies that

$$\int_{\partial V} a_1 \frac{\partial a_1^*}{\partial r}\, dS = -\sum_1^\infty |A_N|^2 (N+1) I_N^2, \tag{A 4}$$

where

$$I_N^2 = \int_{\partial V} [P_N^1(\cos\theta)]^2\, dS$$

and ∂V is the surface of the sphere. Note that the expression (A 4) is real and negative for any set of A_N's.

Hence, since a_1 and $\partial a_1/\partial r$ are continuous across $|\mathbf{r}| = 1$, the same expression holds for the interior fields, and we may write

$$\{a_1 D^2 a_1^*\} = -\left\{|\nabla a_1|^2 + \frac{|a_1|^2}{s^2}\right\} + \int_{\partial V} a_1 \frac{\partial a_1^*}{\partial r}\, dS \tag{A 5}$$

using the divergence theorem, and the surface integral is real and negative. Since also

$$\{b_1 D^2 b_1^*\} = -\{|\nabla b_1|^2 + |b_1|^2/s^2\} \tag{A 6}$$

by similar reasoning, since $b_1 = 0$ on $|\mathbf{r}| = 1$, we may combine (A 2), (A 5) and (A 6) to obtain

$$q_0^2\{|a_1|^2\} + \left\{|\nabla b_1|^2 + \frac{|b_1|^2}{s^2}\right\} - \alpha_0^2\{|b_1|^2\} + iq_0\left[\left\{|\nabla a_1|^2 + \frac{|a_1|^2}{s^2}\right\} - \int_{\partial V} a_1 \frac{\partial a_1^*}{\partial r}\, dS + \{|b_1|^2\}\right] = 0, \tag{A 7}$$

and since the coefficient of iq_0 is real and positive, we conclude that $q_0 = 0$, i.e. that the principle of exchange of stabilities is valid.

Appendix B. Proof of the identity (3.9)

We may write

$$\{\mathbf{B}.\nabla \wedge (\mathbf{U} \wedge \mathbf{B})\} = \{\mathbf{B}.(\mathbf{B}.\nabla\mathbf{U})\} - \{\mathbf{B}.(\mathbf{U}.\nabla\mathbf{B})\}. \tag{B 1}$$

The second term on the right-hand side can be written as $-\{\nabla.(\mathbf{U}(\tfrac{1}{2}\mathbf{B}.\mathbf{B}))\}$ and this is zero by the divergence theorem since $\mathbf{U}.\mathbf{n} = 0$ on $|\mathbf{r}| = 1$. The first term on the right-hand side becomes

$$-\{\mathbf{U}.(\mathbf{B}.\nabla\mathbf{B})\} + \int_{\partial V} (\mathbf{U}.\mathbf{B})\,\mathbf{B}.\mathbf{n}\, dS, \tag{B 2}$$

where ∂V is the sphere $|\mathbf{r}| = 1$, and the first term vanishes from (2.1), since we suppose $E = 0$. The second term becomes

$$\int_{\partial V} B_\theta B_r U_\theta\, dS \tag{B 3}$$

since $B_\phi = U_r = 0$ on $|\mathbf{r}| = 1$. Now, the left-hand side of (B 1) can be written as

$$\{[b\hat{\mathbf{e}}_\phi + \nabla \wedge (a\hat{\mathbf{e}}_\phi)].[d\hat{\mathbf{e}}_\phi + \nabla \wedge (c\hat{\mathbf{e}}_\phi)]\} \tag{B 4}$$

and this can be transformed to give

$$\{bd-cD^2a\}-\int_{\partial V} cB_\theta dS. \tag{B 5}$$

Since $c = U_r B_\theta - U_\theta B_r$, we have

$$\{\mathbf{B}.\nabla \wedge (\mathbf{U} \wedge \mathbf{B})\} \equiv \{bd-cD^2a\}+\int_{\partial V} B_r U_\theta B_\theta dS \tag{B 6}$$

and hence, from (B 3), $\{bd-cD^2a\} \equiv 0$, as required.

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† These papers appear in translation in Roberts & Stix (1971).